

# About the Gamma Function

Notes for Honors Calculus II,  
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## 1 Basic Facts about the Gamma Function

The Gamma function is defined by the improper integral

$$\Gamma(x) = \int_0^{\infty} t^x e^{-t} \frac{dt}{t} .$$

The integral is absolutely convergent for  $x \geq 1$  since

$$t^{x-1} e^{-t} \leq e^{-t/2} , \quad t \gg 1$$

and  $\int_0^{\infty} e^{-t/2} dt$  is convergent. The preceding inequality is valid, in fact, for all  $x$ . But for  $x < 1$  the integrand becomes infinitely large as  $t$  approaches 0 through positive values. Nonetheless, the limit

$$\lim_{r \rightarrow 0+} \int_r^1 t^{x-1} e^{-t} dt$$

exists for  $x > 0$  since

$$t^{x-1} e^{-t} \leq t^{x-1}$$

for  $t > 0$ , and, therefore, the limiting value of the preceding integral is no larger than that of

$$\lim_{r \rightarrow 0+} \int_r^1 t^{x-1} dt = \frac{1}{x} .$$

Hence,  $\Gamma(x)$  is defined by the first formula above for all values  $x > 0$ .

If one integrates by parts the integral

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt ,$$

writing

$$\int_0^{\infty} u dv = u(\infty)v(\infty) - u(0)v(0) - \int_0^{\infty} v du ,$$

with  $dv = e^{-t} dt$  and  $u = t^x$ , one obtains the *functional equation*

$$\Gamma(x+1) = x\Gamma(x) , \quad x > 0 .$$

Obviously,  $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$ , and, therefore,  $\Gamma(2) = 1 \cdot \Gamma(1) = 1$ ,  $\Gamma(3) = 2 \cdot \Gamma(2) = 2!$ ,  $\Gamma(4) = 3\Gamma(3) = 3!$ ,  $\dots$ , and, finally,

$$\Gamma(n+1) = n!$$

for each integer  $n > 0$ .

Thus, the gamma function provides a way of giving a meaning to the “factorial” of any positive real number.

Another reason for interest in the gamma function is its relation to integrals that arise in the study of probability. The graph of the function  $\varphi$  defined by

$$\varphi(x) = e^{-x^2}$$

is the famous “bell-shaped curve” of probability theory. It can be shown that the anti-derivatives of  $\varphi$  are not expressible in terms of elementary functions. On the other hand,

$$\Phi(x) = \int_{-\infty}^x \varphi(t) dt$$

is, by the fundamental theorem of calculus, an anti-derivative of  $\varphi$ , and information about its values is useful. One finds that

$$\Phi(\infty) = \int_{-\infty}^{\infty} e^{-t^2} dt = \Gamma(1/2)$$

by observing that

$$\int_{-\infty}^{\infty} e^{-t^2} dt = 2 \cdot \int_0^{\infty} e^{-t^2} dt ,$$

and that upon making the substitution  $t = u^{1/2}$  in the latter integral, one obtains  $\Gamma(1/2)$ .

To have some idea of the size of  $\Gamma(1/2)$ , it will be useful to consider the qualitative nature of the graph of  $\Gamma(x)$ . For that one wants to know the derivative of  $\Gamma$ .

By definition  $\Gamma(x)$  is an integral (a definite integral with respect to the dummy variable  $t$ ) of a function of  $x$  and  $t$ . Intuition suggests that one ought to be able to find the derivative of  $\Gamma(x)$  by taking the integral (with respect to  $t$ ) of the derivative with respect to  $x$  of the integrand. Unfortunately, there are examples where this fails to be correct; on the other hand, it is correct in most situations where one is inclined to do it. The methods required to justify “differentiation under the integral sign” will be regarded as slightly beyond the scope of this course. A similar stance will be adopted also for differentiation of the sum of a convergent infinite series.

Since

$$\frac{d}{dx} t^x = t^x (\log t) ,$$

one finds

$$\frac{d}{dx} \Gamma(x) = \int_0^\infty t^x (\log t) e^{-t} \frac{dt}{t} ,$$

and, differentiating again,

$$\frac{d^2}{dx^2}\Gamma(x) = \int_0^\infty t^x (\log t)^2 e^{-t} \frac{dt}{t} .$$

One observes that in the integrals for both  $\Gamma$  and the second derivative  $\Gamma''$  the integrand is always positive. Consequently, one has  $\Gamma(x) > 0$  and  $\Gamma''(x) > 0$  for all  $x > 0$ . This means that the derivative  $\Gamma'$  of  $\Gamma$  is a strictly increasing function; one would like to know where it becomes positive.

If one differentiates the functional equation

$$\Gamma(x+1) = x\Gamma(x) , \quad x > 0 ,$$

one finds

$$\psi(x+1) = \frac{1}{x} + \psi(x) , \quad x > 0 ,$$

where

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} ,$$

and, consequently,

$$\psi(n+1) = \psi(1) + \sum_{k=0}^n \frac{1}{k} .$$

Since the harmonic series diverges, its partial sum in the foregoing line approaches  $\infty$  as  $x \rightarrow \infty$ . Inasmuch as  $\Gamma'(x) = \psi(x)\Gamma(x)$ , it is clear that  $\Gamma'$  approaches  $\infty$  as  $x \rightarrow \infty$  since  $\Gamma'$  is steadily increasing and its integer values  $(n-1)!\psi(n)$  approach  $\infty$ . Because  $2 = \Gamma(3) > 1 = \Gamma(2)$ , it follows that  $\Gamma'$  cannot be negative everywhere in the interval  $2 \leq x \leq 3$ , and, therefore, since  $\Gamma'$  is increasing,  $\Gamma'$  must be always positive for  $x \geq 3$ . As a result,  $\Gamma$  must be increasing for  $x \geq 3$ , and, since  $\Gamma(n+1) = n!$ , one sees that  $\Gamma(x)$  approaches  $\infty$  as  $x \rightarrow \infty$ .

It is also the case that  $\Gamma(x)$  approaches  $\infty$  as  $x \rightarrow 0$ . To see the convergence one observes that the integral from 0 to  $\infty$  defining  $\Gamma(x)$  is greater than the integral from 0 to 1 of the same integrand. Since  $e^{-t} \geq 1/e$  for  $0 \leq t \leq 1$ , one has

$$\Gamma(x) > \int_0^1 (1/e)t^{x-1} dt = (1/e) \left[ \frac{t^x}{x} \right]_{t=0}^{t=1} = \frac{1}{ex} .$$

It then follows from the mean value theorem combined with the fact that  $\Gamma'$  always increases that  $\Gamma'(x)$  approaches  $-\infty$  as  $x \rightarrow 0$ .

Hence, there is a unique number  $c > 0$  for which  $\Gamma'(c) = 0$ , and  $\Gamma$  decreases steadily from  $\infty$  to the minimum value  $\Gamma(c)$  as  $x$  varies from 0 to  $c$  and then increases to  $\infty$  as  $x$  varies from  $c$  to  $\infty$ . Since  $\Gamma(1) = 1 = \Gamma(2)$ , the number  $c$  must lie in the interval from 1 to 2 and the minimum value  $\Gamma(c)$  must be less than 1.

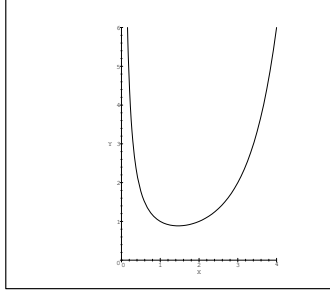


Figure 1: Graph of the Gamma Function

Thus, the graph of  $\Gamma$  (see Figure 1) is concave upward and lies entirely in the first quadrant of the plane. It has the  $y$ -axis as a vertical asymptote. It falls steadily for  $0 < x < c$  to a positive minimum value  $\Gamma(c) < 1$ . For  $x > c$  the graph rises rapidly.

## 2 Product Formulas

It will be recalled, as one may show using l'Hôpital's Rule, that

$$e^{-t} = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^n .$$

From the original formula for  $\Gamma(x)$ , using an interchange of limits that in a more careful exposition would receive further comment, one has

$$\Gamma(x) = \lim_{n \rightarrow \infty} \Gamma(x, n) ,$$

where  $\Gamma(x, n)$  is defined by

$$\Gamma(x, n) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt , \quad n \geq 1 .$$

The substitution in which  $t$  is replaced by  $nt$  leads to the formula

$$\Gamma(x, n) = n^x \int_0^1 t^{x-1} (1-t)^n dt .$$

This integral for  $\Gamma(x, n)$  is amenable to integration by parts. One finds thereby:

$$\Gamma(x, n) = \frac{1}{x} \left(\frac{n}{n-1}\right)^{x+1} \Gamma(x+1, n-1) , \quad n \geq 2 .$$

For the smallest value of  $n$ ,  $n = 1$ , integration by parts yields:

$$\Gamma(x, 1) = \frac{1}{x(x+1)} .$$

Iterating  $n - 1$  times, one obtains:

$$\Gamma(x, n) = n^x \frac{n!}{x(x+1)(x+2) \cdots (x+n)} , \quad n \geq 1 .$$

Thus, one arrives at the formula

$$\Gamma(x) = \lim_{n \rightarrow \infty} n^x \frac{n!}{x(x+1)(x+2) \cdots (x+n)} .$$

This last formula is not exactly in the form of an infinite product

$$\prod_{k=1}^{\infty} p_k = \lim_{n \rightarrow \infty} \prod_{k=1}^n p_k .$$

But a simple trick enables one to maneuver it into such an infinite product. One writes  $n$  as a “collapsing product”:

$$n + 1 = \frac{n+1}{n} \cdot \frac{n}{n-1} \cdot \cdots \cdot \frac{3}{2} \cdot \frac{2}{1}$$

or

$$n + 1 = \prod_{k=1}^n \left( 1 + \frac{1}{k} \right) ,$$

and, taking the  $x$ th power, one has

$$(n+1)^x = \prod_{k=1}^n \left( 1 + \frac{1}{k} \right)^x .$$

Since

$$\lim_{n \rightarrow \infty} \frac{n^x}{(n+1)^x} = 1 ,$$

one may replace the factor  $n^x$  by  $(n+1)^x$  in the last expression above for  $\Gamma(x)$  to obtain

$$\Gamma(x) = \frac{1}{x} \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{\left( 1 + \frac{1}{k} \right)^x}{\left( 1 + \frac{x}{k} \right)} ,$$

or

$$\Gamma(x) = \frac{1}{x} \prod_{k=1}^{\infty} \frac{\left( 1 + \frac{1}{k} \right)^x}{\left( 1 + \frac{x}{k} \right)} .$$

The convergence of this infinite product for  $\Gamma(x)$  when  $x > 0$  is a consequence, through the various maneuvers performed, of the convergence of the original improper integral defining  $\Gamma(x)$  for  $x > 0$ .

It is now possible to represent  $\log \Gamma(x)$  as the sum of an infinite series by taking the logarithm of the infinite product formula. But first it must be noted that

$$\frac{(1+t)^r}{1+rt} > 0 \quad \text{for } t > 0 , \quad r > 0 .$$

Hence, the logarithm of each term in the preceding infinite product is defined when  $x > 0$ .

Taking the logarithm of the infinite product one finds:

$$\log \Gamma(x) = -\log x + \sum_{k=1}^{\infty} u_k(x) ,$$

where

$$u_k(x) = x \log \left( 1 + \frac{1}{k} \right) - \log \left( 1 + \frac{x}{k} \right) .$$

It is, in fact, almost true that this series converges absolutely for *all* real values of  $x$ . The only problem with non-positive values of  $x$  lies in the fact that  $\log(x)$  is meaningful only for  $x > 0$ , and, therefore,  $\log(1 + x/k)$  is meaningful only for  $k > |x|$ . For fixed  $x$ , if one excludes the finite set of terms  $u_k(x)$  for which  $k \leq |x|$ , then the remaining “tail” of the series is meaningful and is absolutely convergent. To see this one applies the “ratio comparison test” which says that an infinite series converges absolutely if the ratio of the absolute value of its general term to the general term of a convergent positive series exists and is finite. For this one may take as the “test series”, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} .$$

Now as  $k$  approaches  $\infty$ ,  $t = 1/k$  approaches 0, and so

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{u_k(x)}{1/k^2} &= \lim_{t \rightarrow 0} \frac{x \log(1+t) - \log(1+xt)}{t^2} \\ &= \lim_{t \rightarrow 0} \frac{\frac{x}{1+t} - \frac{x}{1+xt}}{2t} \\ &= \lim_{t \rightarrow 0} \frac{x[(1+xt) - (1+t)]}{2t(1+t)(1+xt)} \\ &= \frac{x(x-1)}{2} . \end{aligned}$$

Hence, the limit of  $|u_k(x)/k^{-2}|$  is  $|x(x-1)/2|$ , and the series  $\sum u_k(x)$  is absolutely convergent for all real  $x$ . The absolute convergence of this series foreshadows the possibility of defining  $\Gamma(x)$  for all real values of  $x$  other than non-positive integers. This may be done, for example, by using the functional equation

$$\Gamma(x+1) = x\Gamma(x)$$

or

$$\Gamma(x) = \frac{1}{x} \Gamma(x+1)$$

to define  $\Gamma(x)$  for  $-1 < x < 0$  and from there to  $-2 < x < -1$ , etc.

Taking the derivative of the series for  $\log \Gamma(x)$  term-by-term – once again a step that would receive justification in a more careful treatment – and recalling the previous notation  $\psi(x)$  for

the derivative of  $\log \Gamma(x)$ , one obtains

$$\begin{aligned}
\psi(x) + \frac{1}{x} &= \sum_{k=1}^{\infty} \left\{ \log \left( 1 + \frac{1}{k} \right) - \frac{\frac{1}{k}}{\left( 1 + \frac{x}{k} \right)} \right\} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left\{ \log \frac{k+1}{k} - \frac{1}{x+k} \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \log(n+1) - \sum_{k=1}^n \frac{1}{x+k} \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \log(n+1) - \sum_{k=1}^n \frac{1}{k} + \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{x+k} \right) \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \log(n+1) - \sum_{k=1}^n \frac{1}{k} + x \sum_{k=1}^n \frac{1}{k(x+k)} \right\} \\
&= -\gamma + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)} ,
\end{aligned}$$

where  $\gamma$  denotes Euler's constant

$$\gamma = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right) .$$

When  $x = 1$  one has

$$\psi(1) = -1 - \gamma + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} ,$$

and since

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} ,$$

this series collapses and, therefore, is easily seen to sum to 1. Hence,

$$\psi(1) = -\gamma , \quad \psi(2) = \psi(1) + 1/1 = 1 - \gamma .$$

Since  $\Gamma'(x) = \psi(x)\Gamma(x)$ , one finds:

$$\Gamma'(1) = -\gamma ,$$

and

$$\Gamma'(2) = 1 - \gamma .$$

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These course notes were prepared while consulting standard references in the subject, which included those that follow.

## References

- [1] R. Courant, *Differential and Integral Calculus* (2 volumes), English translation by E. J. McShane, Interscience Publishers, New York, 1961.
- [2] E. T. Whittaker & G. N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, 1969.
- [3] David Widder, *Advanced Calculus*, 2nd edition, Prentice Hall, 1961.